XXIV. *On* Tschirnhausen's *Transformation.* *By* Arthur Cayley, *Esq., F.R.S.*

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The memoir of M. Hermite, "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré,"* contains a very important theorem in relation to Tschirnhausen's Transformation of an equation $f(x)=0$ into another of the same degree in y, by means of the substitution $y=\phi x$, where ϕx is a rational and integral function of x. In fact, considering for greater simplicity a quartic equation,

$$(a, b, c, d, e)(x, 1)^4=0,$$

M. Hermite gives to the equation $y=\phi x$ the following form,

$$y=a\mathrm{T}+(ax+b)\mathrm{B}+(ax^2+4bx+6c)\mathrm{C}+(ax^3+4bx^2+6cx+4d)\mathrm{D}$$

(I write B, C, D in the place of his T_0, T_1, T_2), and he shows that the transformed equation in y has the following property: viz., every function of the coefficients which, expressed as a function of a, b, c, d, e, T, B, C, D, does not contain T, is an *invariant*, that is, an invariant of the two quantics

$$(a, b, c, d, e)(\mathrm{X}, \mathrm{Y})^4, \ (\mathrm{B}, \mathrm{C}, \mathrm{D})(\mathrm{Y}, -\mathrm{X})^2.$$

This comes to saying that if T be so determined that in the equation for y the coefficient of the second term (y^3) shall vanish, the other coefficients will be invariants; or if, in the function of y which is equated to zero, we consider y as an absolute constant, the function of y will be an invariant of the two quantics. It is easy to find the value of T; this is in fact given by the equation

$$0=a\mathrm{T}+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D};$$

and we have thence for the value of y,

$$y=(ax+b)\mathrm{B}+(ax^2+4bx+3c)\mathrm{C}+(ax^3+4bx^2+6cx+3d)\mathrm{D};$$

so that for this value of y the function of y which equated to zero gives the transformed equation will be an invariant of the two quantics. It is proper to notice that in the last-mentioned expression for y, all the coefficients except those of the term in x^0, or $b\mathrm{B}+3c\mathrm{C}+3d\mathrm{D}$, are those of the binomial $(1, 1)^4$, whereas the excepted coefficients are those of the binomial $(1, 1)^3$; this suffices to show what the expression for y is in the general case.

I have in the two papers, "Note sur la Transformation de Tschirnhausen,"† and "Deuxième Note sur la Transformation de Tschirnhausen,"† obtained the transformed

* Comptes Rendus, t. xlvi. p. 961 (1858).

† Crelle, t. lviii. pp. 259 and 263 (1861).

equations for the cubic and quartic equations; and by means of a grant from the Government Grant Fund, I have been enabled to procure the calculation, by Messrs. DAVIS and OTTER, under my superintendence, of the transformed equation for the quintic equation. The several results are given in the present memoir; and for greater completeness, I reproduce the demonstration which I have given in the former of the above-mentioned two Notes, of the general property, that the function of y is an invariant. At the end of the memoir I consider the problem of the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$.

Considering for simplicity the foregoing two equations

$$(a, b, c, d, e \between x, 1)^4=0,$$
$$y=(ax+b)\mathrm{B}+(ax^2+4bx+3c)\mathrm{C}+(ax^3+4bx^2+6cx+3d)\mathrm{D};$$

let the second of these be represented by $y=\mathrm{V}$, the transformed equation in y is

$$(y-\mathrm{V}_1)(y-\mathrm{V}_2)(y-\mathrm{V}_3)(y-\mathrm{V}_4)=0,$$

where V_1, V_2, V_3, V_4 are what V becomes upon substituting therein for x the roots x_1, x_2, x_3, x_4 of the quartic equation respectively. Considering y as a constant, the conditions to be satisfied in order that the function in y may be an invariant are that this function shall be reduced to zero by each of the two operators

$$a\partial_b+2b\partial_c+3c\partial_d+4d\partial_e-(\mathrm{D}\partial_{\mathrm{C}}+2\mathrm{C}\partial_{\mathrm{B}}),$$
$$4b\partial_a+3c\partial_b+2d\partial_c+e\partial_d-(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}}):$$

These conditions will be satisfied if each of the factors $y-\mathrm{V}_1$, &c. has the property in question; that is, if $y-\mathrm{V}$, or (what is the same thing) if V, supposing that x denotes therein a root of the quartic equation, is reduced to zero by each of the two operators. Considering the first operator, which for shortness I represent by

$$\Delta-(\mathrm{D}\partial_{\mathrm{C}}+2\mathrm{C}\partial_{\mathrm{B}}),$$

in order to obtain $\Delta\mathrm{V}$ we require the value of Δx. To find it, operating with Δ on the quartic equation, we have

$$(a, b, c, d \between x, 1)^3\Delta x+(a, b, c, d \between x, 1)^3=0,$$

or $\Delta x=-1$. In $\Delta\mathrm{V}$, the part which depends on the variation of Δx then is

$$-a\mathrm{B}+(-2ax-4b)\mathrm{C}+(-3ax^2-8bx-6c)\mathrm{D},$$

and the other part of $\Delta\mathrm{V}$ is at once found to be

$$+a\mathrm{B}+(4ax+6b)\mathrm{C}+(4ax^2+12bx+9c)\mathrm{D};$$

whence, adding,

$$\Delta\mathrm{V}=2(ax+b)\mathrm{C}+(ax^2+4bx+3c)\mathrm{D},$$

and this is precisely equal to

$$(\mathrm{D}\partial_{\mathrm{C}}+2\mathrm{C}\partial_{\mathrm{B}})\mathrm{V};$$

so that V is reduced to zero by the operator $\Delta-(\mathrm{D}\partial_{\mathrm{C}}+2\mathrm{C}\partial_{\mathrm{B}})$.

Similarly, if the second operator is represented by

$$\nabla-(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}}),$$

then we have

$$(a, b, c, d \between x, 1)^3\nabla x+x(b, c, d, e \between x, 1)^3=0,$$

which by means of the equation

$$(a, b, c, d, e \between x, 1)^4=0$$

is reduced to $\nabla x=x^2$. Hence in $\nabla\mathrm{V}$ the part depending on the variation of x is

$$ax^2\mathrm{B}+(2ax^3+4bx^2)\mathrm{C}+(3ax^4+8bx^3+6cx^2)\mathrm{D},$$

and the other part of $\nabla\mathrm{V}$ is at once found to be

$$(4bx+3c)\mathrm{B}+(4bx^2+12cx+6d)\mathrm{C}+(4bx^3+12cx^2+12dx+3e)\mathrm{D};$$

and, adding, the coefficient of D vanishes on account of the quartic equation, and we have

$$\nabla\mathrm{V}=(ax^2+4bx+3c)\mathrm{B}+2(ax^3+4bx^2+6cx+3d)\mathrm{C},$$

which is precisely equal to

$$(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}})\mathrm{V},$$

so that V is reduced to zero by the operator

$$\nabla-(2\mathrm{C}\partial_{\mathrm{D}}+\mathrm{B}\partial_{\mathrm{C}}),$$

which completes the demonstration; and the demonstration in the general case is precisely similar.

In the case of the cubic equation we have

$$(a, b, c, d \between x, 1)^3=0,$$
$$y=(ax+b)\mathrm{B}+(ax^2+3bx+2c)\mathrm{C};$$

and writing the second equation in the form

$$(y-b\mathrm{B}-2c\mathrm{C})+x(\,-a\mathrm{B}-3b\mathrm{C})+x^2(\qquad -a\mathrm{C})=0,$$

multiplying by x and reducing by the cubic equation, we have

$$d\mathrm{C}\qquad +x(y-b\mathrm{B}+\ c\mathrm{C})+x^2(\,-\ a\mathrm{B})=0,$$

and repeating the process,

$$d\mathrm{B}\qquad +x(\ 3c\mathrm{B}+\ d\mathrm{C})+x^2(y+2b\mathrm{B}+c\mathrm{C})=0;$$

or, what is the same thing, we have the system of equations

$$\left(\begin{array}{ccc} y-b\mathrm{B}-2c\mathrm{C}, & -\ a\mathrm{B}-3b\mathrm{C}, & -a\mathrm{C} \\ d\mathrm{C}, & y-\ b\mathrm{B}+\ c\mathrm{C}, & -a\ \mathrm{B} \\ d\mathrm{B}\quad, & 3c\mathrm{B}+\ d\mathrm{C}, & y+2b\mathrm{B}+c\mathrm{C} \end{array}\right| \between 1, x, x^2)=0,$$

and the resulting equation in y is of course that formed by equating to zero the determinant formed out of the matrix in this equation. The developed expression is

$$(1, 0, \mathfrak{C}, \mathfrak{D} \between y, 1)^3=0,$$

where

$\frac{1}{3}\mathfrak{C}=$

	B^2		BC		C^2
ac b^2	$+1$ -1	ad bc	$+1$ -1	bd c^2	$+1$ -1
	± 1		± 1		± 1

$\mathfrak{D}=$

	B^3		B^2C		BC^2		C^3
a^2d abc b^3	$+1$ -3 $+2$	abd ac^2 b^2c	$+3$ -6 $+3$	acd b^2d bc^2	-3 $+6$ -3	ad^2 bcd c^3	-1 $+3$ -2
	± 3		± 6		± 6		± 3

The sum of the coefficients in each column should here and elsewhere in the present memoir be equal to zero, and I have by way of verification annexed to each column the sums ($\pm$ a number) of the positive and negative coefficients. The coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore the function in y, are invariants of the two forms,

$$(a, b, c, d \between X, Y)^3, \quad (B, C \between Y, -X);$$

or in the present case, where there are only two coefficients B, C, the coefficients $\mathfrak{C}$, $\mathfrak{D}$, and therefore also the function in y, are covariants of the single form $(a, b, c, d \between B, C)^3$, considering therein (B, C) as the facients.

It may be remarked that in the present case, assuming the invariance of the function in y, we may obtain the transformed equation in a very simple manner by writing in the first instance $C=0$, this gives

$$(a, b, c, d \between x, 1)^3=0,$$

$$y=(ax+b)B,$$

and thence

$$\frac{1}{a}(a, b, c, d \between y-bB, aB)^3=0;$$

or developing,

$$y^3+3y(ac-b^2)B^2+(a^2d-3abc+2b^3)B^3=0,$$

where the expressions for the coefficients are to be completed by the consideration that these coefficients are covariants of the form $(a, b, c, d \between B, C)^3$. But it is only in the case in hand of a cubic equation that the transformed equation can be obtained in this manner.

In the case of a quartic equation, we have

$$(a, b, c, d, e \between x, 1)^4=0,$$

$$y=(ax+b)B+(ax^2+4bx+3c)C+(ax^3+4bx^2+6cx+3d)D,$$

and these give the system of equations

$$\left(\begin{array}{cccc} y-b\mathrm{B}-3c\mathrm{C}-3d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}-6c\mathrm{D}, & -a\mathrm{C}-4b\mathrm{D}, & -a\mathrm{D} \\ e\mathrm{D}, & y-b\mathrm{B}-3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B}-4b\mathrm{C}, & -a\mathrm{C} \\ e\mathrm{C}, & 4d\mathrm{C}+e\mathrm{D}, & y-b\mathrm{B}+3c\mathrm{C}+d\mathrm{D}, & -a\mathrm{B} \\ e\mathrm{B}, & 4d\mathrm{B}+e\mathrm{C}, & 6c\mathrm{B}+4d\mathrm{C}+e\mathrm{D}, & y+3b\mathrm{B}+3c\mathrm{C}+d\mathrm{D} \end{array}\right\between 1, x, x^2, x^3)=0,$$

and the transformed equation is therefore found by equating to zero the determinant formed out of the matrix contained in this equation.

The developed result, which was obtained by a different process in the 'Deuxième Note' above referred to, is

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\between y, 1)^4=0,$$

where

$\frac{1}{2}\mathfrak{C}=$

	B^2		BC		BD^2	C^2		CD		D^2
ac	+3	ad	+6	ae	+2	+1	be	+6	ce	+3
b^2	−3	bc	−6	bd	−2	+8	cd	−6	d^2	−3
				c^2	.	−9				
	±3		±6		±2	±9		±6		±3

$\frac{1}{4}\mathfrak{D}=$

	B^3		B^2C		B^2D	BC^2		BCD	C^3		BD^2	C^2D		CD^2		D^3
							ac^2									
a^2d	+1	a^2e	+1	abe	+1	+ 4	ad^2	−6	−4	ade	−1	− 4	ae^2	−1	be^2	−1
abc	−3	abd	+2	acd	−3	−12	b^2e	+6	+4	bcd	+3	+12	bde	−2	cde	+3
b^3	+2	ac^2	−9	b^2d	+2	+ 8	bcd			bd^2	−2	− 8	c^2e	+9	d^3	−2
		b^2c	+6	bc^2			c^3			c^2d			cd^2	−6		
	±3		±9		±3	±12		±6	±4		±3	±12		±9		±3

and

$\mathfrak{E}=$

	B^4		B^3C		B^3D	B^2C^2		B^2CD	BC^3		B^2D^2	BC^2D	C^4
a^3e	+1	a^2be	+ 8	a^2ce	+12	− 6	a^2de	∽	− 4	a^2e^2	+ 2	− 4	+ 1
a^2bd	−4	a^2cd	−12	a^2d^2	−12	∽	$abce$	+60	− 12	$abde$	−16	+ 20	− 16
a^2c^2	∽	ab^2d	−20	ab^2e	− 8	+30	abd^2	−72	+ 16	ac^2e	+36	+ 36	− 18
ab^2c	+6	abc^2	+36	$abcd$	+12	−48	ac^2d	+36	+ 36	acd^2	−18	∽	+ 48
b^4	−3	b^3c	− 12	ac^3	∽	+54	b^3e	−36	+ 48	b^2ce	−18	∽	+ 48
				b^3d	− 4	−48	b^2cd	+12	−192	b^2d^2	+14	−160	∽
				b^2c^2	.	+18	bc^3	.	+108	bc^2d	.	+108	−144
										c^4	.	.	+ 81
	±7		±44		±24	±102		±108	±208		±52	±164	±178

	BCD^2	C^3D		BD^3	C^2D^2		CD^3		D^4
abe^2	∽	− 4	ace^2	+12	− 6	ade^2	+ 8	ae^3	+1
$acde$	+60	− 12	ad^2e	− 8	+30	bce^2	−12	bde^2	−4
ad^3	−36	+ 48	b^2e^2	−12	∽	bd^2e	−20	c^2e^2	∽
b^2de	−72	+ 16	$bcde$	+12	−48	c^2de	+36	cd^2e	+6
bc^2e	+36	+ 36	bd^3	− 4	−48	cd^3	−12	d^4	−3
bcd^2	+12	−192	c^3e	.	+54				
c^3d	.	+108	c^2d^2	.	+18				
	±108	±208		±24	±102		±44		±7

I write

$$U'=aB^2+4bBC+c(2BD+4C^2)+4dCD+eD^2,$$
$$H'=(ac-b^2)B^2+2(ad-bc)BC+(ae-2bd+c^2)BD+4(bd-c^2)C^2$$
$$+2(be-cd)CD+(ce-d^2)D^2;$$

and I represent by Φ' the expression which has just been found for $\frac{1}{4}\mathfrak{D}$. These functions, U', H', Φ', are invariants of the two forms

$$(a, b, c, d, e \between X, Y)^4, \quad (B, C, D \between Y, -X)^2;$$

we have, moreover, the invariants

$$ae-4bd+3c^2, \; ace-ad^2-b^2e+2bcd-c^3,$$

which I represent as usual by I, J, and the invariant $BD-C^2$, which I represent by Θ'. This being so, we have

$$\mathfrak{C}=6H'-2I\Theta',$$
$$\mathfrak{D}=4\Phi',$$
$$\mathfrak{E}=IU'^2-3H'^2+I^2\Theta'^2+12J\Theta'U'+2I\Theta'H',$$

the last of which may be verified as follows:—viz. writing $a=e=1$, $b=d=0$, $c=\theta$, it becomes

$$\begin{aligned}&(1+3\theta^2)\{B^2+\theta(2BD+4C^2)+D^2\}^2\\ &-3\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}^2\\ &+(1+3\theta^2)^2(BD-C^2)^2\\ &+12(\theta-\theta^3)(BD-C^2)\{\theta B^2+(1+\theta^2)BD-4\theta^2C^2+\theta D^2\}\\ =\;& B^4\\ &+B^3C \quad (12\theta)\\ &+B^2C^2 \quad (-6\theta+54\theta^3)\\ &+B^2D^2 \quad (2+36\theta^2)\\ &+BC^2D \quad (-4+36\theta^2)\\ &+B^4 \quad (1-18\theta^2+81\theta^4)\\ &+C^2D^2 \quad (-6\theta+54\theta^3)\\ &+C^3D \quad (12\theta)\\ &+D^4,\end{aligned}$$

which is an identical equation.

The expression for the invariant I (quadrinvariant) of the function $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E} \between y, 1)^4$ is $\mathfrak{E}+3(\frac{1}{6}\mathfrak{C})^2$, or $\mathfrak{E}+3(H'-\frac{1}{3}I\Theta')^2$, viz. it is

$$\begin{aligned}&IU'^2-3H'^2+I^2\Theta'^2+12J\Theta'U'+2I\Theta'H'\\ &\quad +3H'^2+\tfrac{1}{3}I^2\Theta'^2 \qquad -2I\Theta'H',\end{aligned}$$

or, finally, it is

$$IU'^2+\tfrac{4}{3}I^2\Theta'^2+12J\Theta'U',$$

which is equal to

$$\frac{1}{\mathrm{I}}\left[(\mathrm{IU}'+6\mathrm{J}'\Theta')^2+\tfrac{4}{3}(\mathrm{I}^3-27\mathrm{J}^2)\Theta'^2\right];$$

so that the condition in order that this invariant may be equal to zero is

$$\mathrm{IU}'+\left[6\mathrm{J}\pm2\sqrt{-\tfrac{1}{3}(\mathrm{I}^3-27\mathrm{J}^2)}\right]\Theta'=0,$$

which agrees with a result of M. HERMITE's.

There should, I think, be an identical equation of the form

$$\mathrm{JU}'^2-\mathrm{IU}'^2\mathrm{H}'+4\mathrm{H}'^3+\mathrm{M}\Theta'=-\Phi'^2,$$

which would serve to express the square of the invariant Φ' in terms of the other invariants U', H', Θ', I, J; but assuming that such an equation exists, the form of the factor M remains to be ascertained. The invariant J (cubinvariant) of the form $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\between y, 1)^4$ contains Φ'^2, and it would be necessary to make use of the identical equation just referred to in order to reduce it to its simplest form; and (this being so) I have not sought for the expression of the cubinvariant of $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}\between y, 1)^4$.

For the quintic we have the equations

$$\begin{aligned}
&(a, b, c, d, e, f\between x, 1)^5=0,\\
y=\ &(ax+b)\mathrm{B}\\
&+(ax^2+5bx+4c)\mathrm{C}\\
&+(ax^3+5bx^2+10cx+6d)\mathrm{D}\\
&+(ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E},
\end{aligned}$$

and this leads to the system of equations

$$\left(\begin{array}{ccccc}
y-b\mathrm{B}-4c\mathrm{C}-6d\mathrm{D}-4e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}-10d\mathrm{E}, & -a\mathrm{C}-5b\mathrm{D}-10c\mathrm{E}, & -a\mathrm{D}-5b\mathrm{E}, & -a\mathrm{E}\\
f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}-6c\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}-10c\mathrm{D}, & -a\mathrm{C}-5b\mathrm{D}, & -a\mathrm{D}\\
f\mathrm{D}, & 5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}-4c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}-5b\mathrm{C}, & -a\mathrm{C}\\
f\mathrm{C}, & 5e\mathrm{C}+f\mathrm{D}, & 10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E}, & y-b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}, & -a\mathrm{B}\\
f\mathrm{B}, & 5e\mathrm{B}+f\mathrm{C}, & 10d\mathrm{B}+5e\mathrm{C}+f\mathrm{D}, & 10c\mathrm{B}+10d\mathrm{C}+5e\mathrm{D}+f\mathrm{E}, & y+4b\mathrm{B}+6c\mathrm{C}+4d\mathrm{D}+e\mathrm{E}
\end{array}\between 1, x, x^2, x^3, x^4\right)=0,$$

and the transformed equation is obtained by equating to zero the determinant formed out of the matrix contained in this equation.

The determinant in question was calculated by the formula

	Div.
$\square = -12.345$	1
$+13.245$	2
-14.235	3
$+15.234$	4
-23.145	5
$+24.135$	6
-25.134	7
-34.125	8
$+35.124$	9
-45.123	10,

where the duadic symbols refer to the first and fifth columns, viz. 12 is the determinant formed out of the lines 1 and 2 of these columns, and so for the other like symbols; and the triadic symbols refer to the second, third, and fourth columns, viz. 345 is the determinant formed out of the lines 3, 4, 5 of these columns, and so for the other like symbols.

The ten divisions were separately calculated. It is to be noticed that these divisions other than 4 and 6 correspond to each other in pairs, while each of the divisions 4 and 6 corresponds to itself, as thus:

Div. 1,	-10
2,	$-\ 9$
3,	$-\ 7$
5,	$-\ 8$
4,	$-\ 4$
6,	$-\ 6$,

viz. if in the place of

$$y;\ a,\ b,\ c,\ d,\ e, f;\ \mathrm{B,\ C,\ D,\ E},$$

we write

$$-y;\ f,\ e,\ d,\ c,\ b,\ a;\ \mathrm{E,\ D,\ C,\ B},$$

then division 1 becomes division 10 with its sign reversed, and so for divisions 2 and 9, 3 and 7, 5 and 8; while each of the divisions 4 and 6 is unaltered, except that the sign is reversed. But the corresponding divisions were each of them calculated, and the property in question was used as a verification. Another very convenient verification, which was employed for the several divisions, was obtained by putting

$$a=b=c=d=e=f=\mathrm{B}=\mathrm{C}=\mathrm{D}=\mathrm{E}=1,$$

upon which supposition the determinant becomes

$$\begin{vmatrix} y-15, & -26, & -16, & -6, & -1 \\ 1, & y-10, & -16, & -6, & -1 \\ 1, & 6, & y\ , & -6, & -1 \\ 1, & 6, & 16, & y+10, & -1 \\ 1, & 6, & 16, & 16, & y+15 \end{vmatrix}$$

and the values of the ten divisions respectively are

y^5,	y^4,	y^3,	y^2,	y,	1	
		6,	−288,	+ 4608,	−24576	1
		16,	−576,	+ 6144,	−16384	2
		26,	−544,	+ 3584,	−24576	3
1,	0,	−96,	0 ,	−28672,	0	4
				0 ,	0	5
				0 ,	0	6
		26,	+544,	+ 3584,	+24576	7
				0 ,	0	8
		16,	+576,	+ 6144,	+16384	9
		6,	+288,	+ 4608,	+24576	10
1,	0,	0,	0 ,	0 ,	0	

A verification similar to this was in fact employed at each step of the calculation of a division: viz. in forming a product such as $(\lambda X+\mu Y+\&c.)(\lambda' X+\mu' Y+\&c.)$, where λ, μ, &c., λ', μ', .. &c. are numerical coefficients, and X, Y, &c. are monomial products of a, b, c, d, e, f and B, C, D, E, the sum of the numerical coefficients of the product is $(\lambda+\mu+\&c.)(\lambda'+\mu'+\&c.)$.

It was of course necessary to employ such verifications, as a test of the correctness of the several divisions, before proceeding to collect them together, but the collection itself affords an exceedingly good ultimate verification. The following is an exemplification: the terms in y which involve the product BCDE are obtained by the collection of the corresponding terms in the ten divisions, as follows:

			1	2	3	4	5	6	7	8	9	10	
y	BCDE. − 5	a^2f^2	+ 1	− 2	− 2	+ 1	+ 1	− 2	− 2	+ 1	− 2	+ 1	
	+ 30	$abef$	+ 49	− 16	− 20	+ 4	− 25	+50	− 20	− 25	− 16	+ 49	
	+ 980	$acdf$	+ 80	+200	+184	− 148	+100		+184	+100	+200	+ 80	
	− 280	ace^2	+ 80	.	+ 80	− 440							
	− 180	ad^2e	.	− 60	− 60	− 60							
	− 280	b^2df	.	.		− 440	.		+ 80	.	.	+ 80	
	− 825	b^2e^2	.	.		− 825							
	− 180	bc^2f	.	.		− 60	.		− 60	.	− 60		
	+ 740	$bcde$	.	.		+ 740							
		bd^3											
		c^3e											
		c^2d^2											
	±1750=0		+210	+122	+182	−1228	+ 76	+48	+182	+ 76	+122	+210	=±1228

where it may be remarked that the greater part, but not all, of the component coefficients are divisible by 5. I soon observed in the process of summing the ten divisions that all the resulting coefficients should be divisible by 5 (the only exception is as to the terms in y^0 which contain B^5, C^5, D^5, and E^5 respectively), and the circumstance that they are so in each particular instance is as far as it goes a verification, which, however, only applies to those of the component coefficients which are not themselves divisible by 5. But it was known *à priori* (I will presently show how this is so) that the sum of the resulting coefficients should be equal to zero, and that they are so in fact is a verification as to *all* the coefficients. The foregoing specimen term BCDE is one which remains unaltered when B, C, D, E are changed into E, D, C, B; and on making the further change a, b, c, d, e, f into f, e, d, c, b, a, the coefficient of BCDE remains, as it should do, unaltered; this is a verification of the coefficients of the terms ace^2, b^2df; ad^2e, bc^2f, which are respectively interchanged by the substitution in question, but not of the other terms a^2f^2, $abef$, $acdf$, b^2e^2, $bcdf$, which are respectively unaltered by the substitution. I did *not* employ what would have been another convenient verification of the several divisions, viz. the comparison of their values on putting therein $a=b=c=d=e=f=1$, with the corresponding values as calculated independently from the determinant

$$\begin{vmatrix} y-B-4C-6D-E, & -B-5C-10D-10E, & -C-5D-10E, & D-5E, & -E \\ E, & y-B-4C-6D+E, & -B-5C-10D\;, & -C-5D\;, & -D \\ D\;, & 5D+E, & y-B-4C+4D+E, & -B-5C\;, & -C \\ C\;, & 5C+D\;, & 10C+5D+E, & y-B+6C+4D+E, & -B \\ B\;, & 5B+C\;, & 10B+5C+D\;, & 10B+10C+5D+E, & y+4B+6C+4D+E \end{vmatrix}$$

The calculation of the ten divisions of this determinant would however itself have been a work of some labour.

The last-mentioned determinant is $=y^5$; in fact, equating it to zero, we have the transformed equation corresponding to the system of equations

$$(1,1,1,1,1,1\between x,1)^5=0,$$

$$y=(x+1)B+(x^2+5x+4)C+(x^3+5x^2+10x+6)D+(x^4+5x^3+10x^2+10x+4)E.$$

But the first of these equations is $(x+1)^5=0$, and the second is

$$y=(x+1)\{B+(x+4)C+(x^2+4x+6)D+(x^3+4x^2+6x+4)E\},$$

so that for each of the five equal roots $x=-1$, we have $y=0$, or the transformed equation in y is $y^5=0$.

And since upon writing $a=b=c=d=e=f=1$ the transformed equation becomes $y^5=0$, it is clear that in the coefficient of any monomial product of B, C, D, E, the sum of the numerical coefficients of the several monomial products of a, b, c, d, e, f must be $=0$, which is the property above referred to as affording a verification of the calculated expression of the transformed equation.

The final result is that the equations

$$(a, b, c, d, e, f \between x, 1)^5=0,$$

$$\begin{aligned} y= & \ (ax+b)\mathrm{B} \\ & +(ax^2+5bx+4c)\mathrm{C} \\ & +(ax^3+5bx^2+10cx+6d)\mathrm{D} \\ & +(ax^4+5bx^3+10cx^2+10dx+4e)\mathrm{E} \end{aligned}$$

give for the transformed equation in y

$$(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0,$$

where

$\frac{1}{5}\mathfrak{C}=$

	B²		BC		BD	C²		BE	CD		CE	D²		DE		E²
ac	+2	ad	+6	ae	+4	+ 2	af	+1	+ 1	bf	+4	+ 2	cf	+6	df	+2
b^2	−2	bc	−6	bd	−4	+10	be	−1	+15	ce	−4	+10	de	−6	e^2	−2
				c^2	.	−12	cd	.	−16	d^2	.	−12				
	±2		±6		±4	±12		±1	±16		±4	±12		±6		±2

$\frac{1}{5}\mathfrak{D}=$

	B³		B²C		B²D	BC²		B²E	BCD	C³		BCE	BD²	C²D
a^2d	+2	a^2e	+ 4	a^2f	+1	+ 1	abf	+1	+10	+ 3	acf	∽	+ 6	+ 4
abc	−6	abd	+ 2	abe	+3	+19	ace	−4	− 8	− 4	ade	−12	−22	− 46
b^3	+4	ac^2	−24	acd	∽	−52	ad^2	∽	−48	−20	b^2f	+ 8	+ 4	+ 20
		b^2c	+18	b^2d	−4	+20	b^2e	+3	+30	+25	bce	+ 4	+20	+ 70
				bc^2	.	+12	bcd	.	+16	−20	bd^2	.	− 8	− 80
							c^3	.	.	+16	c^2d			+ 32
	±6		±24		±4	±52		±4	±56	±44		±12	±30	±126

	BDE	C²E	CD²		BE²	CDE	D³		CE²	D²E		DE²		E³
adf	∽	− 6	− 4	aef	−1	−10	− 3	af^2	−1	− 1	bf^2	− 4	cf^2	−2
ae^2	− 8	− 4	− 20	bdf	+4	+ 8	+ 4	bef	−3	−19	cef	− 2	def	+6
bcf	+12	+22	+ 46	be^2	−3	−30	−25	cdf	∽	+52	d^2f	+24	e^3	−4
bde	− 4	−20	− 70	c^2f		+48	+20	ce^2	+4	−20	de^2	−18		
c^2e		+ 8	+ 80	cde		−16	+20	d^2e		−12				
cd^2			− 32	d^3			−16							
	±12	±30	±126		±4	±56	±44		±4	±52		±24		±6

$\frac{1}{5}\mathfrak{E}=$

	B^4
a^3e	+1
a^2bd	−4
a^2c^2	∞
ab^2c	+6
b^4	−3
	±7

	B^3C
a^3f	+ 1
a^2be	+ 7
a^2cd	−16
ab^2d	−22
abc^2	+48
b^3c	−18
	±56

	B^3D	B^2C^2
a^2bf	+ 2	+ 9
a^2ce	+20	− 22
a^2d^2	−24	∞
ab^2e	−18	+ 31
$abcd$	+32	− 56
ac^3	∞	+ 96
b^3d	−12	− 70
b^2c^2		+ 12
	±54	±148

	B^3E	B^2CD	BC^3
a^2cf	+ 6	+ 10	− 2
a^2de	− 6	− 18	− 10
ab^2f	− 5	+ 17	+ 29
$abce$	+ 8	+ 96	− 92
abd^2	∞	−144	+ 40
ac^2d	∞	+128	+128
b^3e	− 3	−105	+ 75
b^2cd	.	+ 16	−360
bc^3	.	.	+192
	±14	±267	±464

	B^2CE	B^2D^2	BC^2D	C^4
a^2df	+12	− 6	∞	− 4
a^2e^2	−12	+ 10	− 20	+ 5
$abcf$	+28	+ 38	+ 48	+ 6
$abde$	−36	− 86	− 8	− 50
ac^2e	+32	+100	+ 72	− 64
acd^2	∞	− 24	+144	+160
b^3f	−28	− 14	+ 60	+ 25
b^2ce	+ 4	− 70	−240	+ 50
b^2d^2	.	+ 52	−440	∞
bc^2d	.	.	+384	−320
c^4	.	.	.	+192
	±76	±200	±708	±438

	B^2DE	BC^2E	BCD^2	C^3D
a^2ef	+ 5	− 5	− 7	+ 1
$abdf$	− 8	+ 92	− 8	− 36
abe^2	−29	− 67	+ 15	− 25
ac^2f	+60	+ 12	+ 72	− 20
$acde$	−12	+ 36	+196	+ 28
ad^3	∞	∞	− 72	+240
b^2cf	−42	− 54	+ 78	+190
b^2de	+26	−110	−510	− 50
bc^2e	.	+ 96	− 60	−280
bcd^2	.	.	+296	−560
c^3d	.	.	.	+512
	±91	±236	±657	±971

	B^2E^2	$BCDE$	BD^3	C^3E	C^2D^2
a^2f^2	+ 1	− 1	− 1	− 1	+ 1
$abef$	− 5	+ 6	− 4	− 4	− 25
$acdf$	+20	+196	+ 28	+ 28	− 80
ace^2	−14	− 56	+100	− 28	+ 10
ad^2e	∞	− 36	− 68	+ 60	+ 336
b^2df	−14	− 56	− 28	+100	+ 10
b^2e^2	+12	−165	− 75	− 75	∞
bc^2f	.	− 36	+ 60	− 68	+ 336
$bcde$	.	+148	−140	−140	− 940
bd^3	.	.	+128	.	− 120
c^3e	.	.	.	+128	− 120
c^2d^2	.	.	.	.	+ 592
	±33	±350	±316	±316	±1285

	BCE^2	BD^2E	C^2DE	CD^3
abf^2	+ 5	− 5	− 7	+ 1
$acef$	− 8	+ 92	− 8	− 36
ad^2f	+60	+ 12	+ 72	− 20
ade^2	−42	− 54	+ 78	+190
b^2ef	−29	− 67	+ 15	− 25
$bcdf$	−12	+ 36	+196	+ 28
bce^2	+26	−110	−510	− 50
bd^2e	.	+ 96	− 60	−280
c^3f	.	.	− 72	+240
c^2de	.	.	+296	−560
cd^3	.	.	.	+512
	±91	±236	±657	±971

	BDE^2	C^2E^2	CD^2E	D^4
acf^2	+12	− 6	∞	− 4
$adef$	+28	+ 38	+ 48	+ 6
ae^3	−28	− 14	+ 60	+ 25
b^2f^2	−12	+ 10	− 20	+ 5
$bcef$	−36	− 86	− 8	− 50
bd^2f	+32	+100	+ 72	− 64
bde^2	+ 4	− 70	−240	+ 50
c^2df	.	− 24	+144	+160
c^2e^2	.	+ 52	−440	∞
cd^2e	.	.	+384	−320
d^4	.	.	.	+192
	±76	±200	±708	±438

	BE^3	CDE^2	D^3E
adf^2	+ 6	+ 10	− 2
ae^2f	− 5	+ 17	+ 29
bcf^2	− 6	− 18	− 10
$bdef$	+ 8	+ 96	− 92
be^3	− 3	−105	+ 75
c^2ef	.	−144	+ 40
cd^2f	.	+128	+128
cde^2	.	+ 16	−360
d^3e	.	.	+192
	±14	±267	±464

	CE^3	D^2E^2
aef^2	+ 2	+ 9
bdf^2	+20	− 22
be^2f	−18	+ 31
c^2f^2	−24	∞
$cdef$	+32	− 56
ce^3	−12	− 70
d^3f	.	+ 96
d^2e^2	.	+ 12
	±54	±148

	DE^3
af^3	+ 1
bef^2	+ 7
cdf^2	−16
ce^2f	−22
d^2ef	+48
de^3	−18
	±56

	E^4
bf^3	+1
cef^2	−4
d^2f^2	∞
de^2f	+6
e^4	−3
	±7

$\mathfrak{F}$ =

	B^5		B^4C		B^4D	B^3C^2		B^4E	B^3CD	B^2C^3
a^4f	+ 1	a^3bf	+ 15	a^3cf	+ 30	− 10	a^3df	+ 20	ꝏ	− 10
a^3be	− 5	a^3ce	− 20	a^3de	− 30	ꝏ	a^3e^2	− 20	ꝏ	ꝏ
a^3cd	ꝏ	a^3d^2	ꝏ	a^2b^2f	− 15	+ 100	a^2bcf	− 30	+ 320	− 60
a^2b^2d	+ 10	a^2b^2e	− 55	a^2bce	− 100	− 190	a^2bde	+ 30	− 360	+ 50
a^2bc^2	ꝏ	a^2bcd	+ 80	a^2bd^2	+ 120	ꝏ	a^2c^2e	ꝏ	− 400	+ 120
ab^3c	− 10	a^2c^3	ꝏ	a^2c^2d	ꝏ	+ 160	a^2cd^2	ꝏ	+ 480	ꝏ
b^5	+ 4	ab^3d	+ 70	ab^3e	+ 55	− 260	ab^3f	+ 15	− 140	+ 340
		ab^2c^2	− 120	ab^2cd	− 80	+ 540	ab^2ce	− 20	− 440	− 1020
		b^4c	+ 30	abc^3	ꝏ	− 480	ab^2d^2	ꝏ	+ 960	− 100
				b^4d	+ 20	+ 200	abc^2d	ꝏ	− 640	+ 960
				b^3c^2	.	− 60	ac^4	ꝏ	ꝏ	− 640
							b^4e	+ 5	+ 300	− 500
							b^3cd	.	− 80	+ 1900
							b^2c^3	.	.	− 1040
	±15		±195		±225	±1000		±70	±2060	±3370

	B^3CE	B^3D^2	B^2C^2D	BC^4		B^3DE	B^2C^2E	B^2CD^2	BC^3D	C^5
a^3ef	ꝏ	+ 10	− 20	+ 5	a^3f^2	+ 5	− 5	− 5	+ 5	− 1
a^2bdf	+ 240	− 40	+ 20	− 80	a^2bef	− 30	+ 30	+ 60	− 130	+ 25
a^2be^2	− 240	− 50	+ 100	− 25	a^2cdf	+ 220	− 40	+ 380	− 380	+ 80
a^2c^2f	− 120	+ 300	+ 20	− 40	a^2ce^2	− 400	+ 40	− 200	+ 400	− 100
a^2cde	+ 120	− 600	+ 60	+ 200	a^2d^2e	+ 180	ꝏ	− 180	+ 300	ꝏ
a^2d^3	ꝏ	+ 360	ꝏ	ꝏ	ab^2df	− 40	+ 1070	− 640	+ 380	− 250
ab^2cf	− 140	− 220	+ 1190	+ 20	ab^2e^2	+ 185	− 1145	+ 25	+ 125	ꝏ
ab^2de	+ 240	+ 540	− 1980	+ 250	abc^2f	− 300	− 940	+ 1640	+ 940	− 360
abc^2e	− 160	− 500	− 3160	+ 320	$abcde$	+ 60	+ 1020	− 4380	− 1140	+ 1000
$abcd^2$	ꝏ	+ 120	+ 4080	− 800	abd^3	ꝏ	ꝏ	+ 3960	− 1200	ꝏ
ac^3d	ꝏ	ꝏ	− 1280	− 320	ac^3e	ꝏ	− 320	− 2000	− 2080	+ 960
b^4f	+ 80	+ 40	− 400	+ 500	ac^2d^2	ꝏ	ꝏ	+ 480	+ 960	− 1600
b^3ce	− 20	+ 200	+ 850	− 2750	b^3cf	+ 120	+ 160	− 620	+ 800	+ 750
b^3d^2	.	− 160	+ 2600	ꝏ	b^3de	− 80	+ 650	+ 2900	− 3500	ꝏ
b^2c^2d	.	.	− 2080	+ 5600	b^2c^2e	.	− 520	− 100	− 2600	− 300
bc^4	.	.	.	− 2880	b^2cd^2	.	.	− 1320	+ 14800	ꝏ
					bc^3d	.	.	.	− 7680	+ 4800
					c^5	.	.	.	.	− 2304
	±680	±1570	±8920	±6895		±810	±2970	±9445	±18710	±7615

	B^3E^2	B^2CDE	B^2D^3	BC^3E	BC^2D	C^4D		B^2CE^2	B^2D^2E	BC^2DE	BCD^3	C^4E	C^3D^2
a^2bf^2	+ 5	+ 15	− 10	− 25	ꝏ	+ 5	a^2cf^2	− 50	+ 70	+ 10	− 50	− 10	+ 20
a^2cef	− 70	− 60	+ 200	+ 80	− 210	+ 30	a^2def	+ 110	+ 80	− 340	+ 160	+ 130	− 30
a^2d^2f	+ 100	+ 240	+ 100	− 200	− 180	+ 120	a^2e^3	− 80	− 200	+ 400	ꝏ	− 100	ꝏ
a^2de^2	− 40	− 240	− 300	+ 200	+ 600	− 150	ab^2f^2	+ 80	− 70	− 40	+ 20	− 25	ꝏ
ab^2ef	+ 30	− 30	− 90	+ 70	ꝏ	− 125	$abcef$	− 520	− 460	+ 620	+ 380	− 20	− 500
$abcdf$	− 100	+ 1380	− 640	+ 260	+ 1920	− 1180	abd^2f	+ 700	+ 460	+ 1240	− 760	− 1000	+ 540
$abce^2$	+ 70	− 2820	− 500	− 360	+ 450	+ 500	$abde^2$	− 190	+ 620	− 2140	− 200	+ 1000	+ 750
abd^2e	ꝏ	+ 1980	+ 1540	− 300	− 3180	+ 1500	ac^2df	− 400	+ 200	− 1720	+ 2840	− 80	− 880
ac^3f	ꝏ	− 1200	+ 1000	− 560	+ 1080	− 240	ac^2e^2	+ 280	− 2000	− 2240	− 2000	+ 880	+ 1400
ac^2de	ꝏ	+ 240	− 3000	+ 240	− 8160	+ 3120	acd^2e	ꝏ	+ 1080	+ 2520	− 6440	− 1200	− 120
acd^3	ꝏ	ꝏ	+ 1440	ꝏ	+ 5040	− 4800	ad^4	ꝏ	ꝏ	ꝏ	+ 4320	ꝏ	− 3600
b^3df	+ 40	+ 240	+ 120	+ 2000	− 1800	+ 500	b^3ef	+ 160	+ 430	− 200	− 500	+ 125	ꝏ
b^3e^2	− 35	+ 975	+ 500	− 2125	ꝏ	ꝏ	b^2cdf	− 20	− 440	+ 5320	− 2240	+ 3000	− 700
b^2c^2f	.	− 60	− 400	− 1060	+ 1620	+ 2100	b^2ce^2	− 70	+ 650	− 2950	+ 2750	− 3500	ꝏ
b^2cde	.	− 660	+ 600	+ 3700	+ 2700	− 6500	b^2d^2e	.	− 420	+ 4800	+ 7400	ꝏ	− 3000
b^2d^3	.	.	− 560	ꝏ	+ 9600	ꝏ	bc^3f	.	.	− 3240	+ 1800	− 1680	+ 3960
bc^3e	.	.	.	− 1920	− 5400	− 4800	bc^2de	.	.	− 2040	− 10200	+ 4400	− 5600
bc^2d^2	.	.	.	.	− 4080	+ 17600	bcd^3	.	.	.	+ 2720	ꝏ	+ 20400
c^4d	.	.	.	.	.	− 7680	c^4e	.	.	.	.	− 1920	− 7200
							c^3d^2	.	.	.	.	.	− 5440
	±245	±5070	±5500	±6550	±23010	±25475		±1330	±3590	±14910	±22390	±9535	±27070

	B^2DE^2	BC^2E^2	BCD^2E	BD^4	C^3DE	C^2D^3
a^2df^2	+ 50	− 70	− 10	+ 10	+ 50	− 20
a^2e^2f	− 80	+ 70	+ 40	+ 25	− 20	∽
$abcf^2$	− 110	− 80	+ 340	− 130	− 160	+ 30
$abdef$	+ 520	+ 460	− 620	+ 20	− 380	+ 500
abe^3	− 160	− 430	+ 200	− 125	+ 500	∽
ac^2ef	− 700	− 460	− 1240	+ 1000	+ 760	− 540
acd^2f	+ 400	− 200	+ 1720	+ 80	− 2840	+ 880
$acde^2$	+ 20	+ 440	− 5320	− 3000	+ 2240	+ 700
ad^3e	∽	∽	+ 3240	+ 1680	− 1800	− 3960
b^3f^2	+ 80	+ 200	− 400	+ 100	∽	∽
b^2cef	+ 190	− 620	+ 2140	− 1000	+ 200	− 750
b^2d^2f	− 280	+ 2000	+ 2240	− 880	+ 2000	− 1400
b^2de^2	+ 70	− 650	+ 2950	+ 3500	− 2750	∽
bc^2df	.	− 1080	− 2520	+ 1200	+ 6440	+ 120
bc^2e^2	.	+ 420	− 4800	∽	− 7400	+ 3000
bcd^2e	.	.	+ 2040	− 4400	+ 10200	+ 5600
bd^4	.	.	.	+ 1920	∽	+ 7200
c^4f	.	.	.	.	− 4320	+ 3600
c^3de	.	.	.	.	− 2720	− 20400
c^2d^3	.	.	.	.	.	+ 5440
	±1330	±3590	±14910	±9535	±22390	±27070

	B^2E^3	$BCDE^2$	BD^3E	C^3E^2	C^2D^2E	CD^4
a^2ef^2	− 5	− 15	+ 25	+ 10	∽	− 5
$abdf^2$	+ 70	+ 60	− 80	− 200	+ 210	− 30
abe^2f	− 30	+ 30	− 70	+ 90	∽	+ 125
ac^2f^2	− 100	− 240	+ 200	− 100	+ 180	− 120
$acdef$	+ 100	− 1380	− 260	+ 640	− 1920	+ 1180
ace^3	− 40	− 240	− 2000	− 120	+ 1800	− 500
ad^3f	∽	+ 1200	+ 560	− 1000	− 1080	+ 240
ad^2e^2	∽	+ 60	+ 1060	+ 400	− 1620	− 2100
b^2cf^2	+ 40	+ 240	− 200	+ 300	− 600	− 150
b^2def	− 70	+ 2820	+ 360	+ 500	− 450	− 500
b^2e^3	+ 35	− 975	+ 2125	− 500	∽	∽
bc^2ef	.	− 1980	+ 300	− 1540	+ 3180	+ 1500
bcd^2f	.	− 240	− 240	+ 3000	+ 8160	− 3120
$bcde^2$	.	+ 660	− 3700	− 600	− 2700	+ 6500
bd^3e	.	.	+ 1920	∽	+ 5400	+ 4800
c^3df	.	.	.	− 1440	− 5040	+ 4800
c^3e^2	.	.	.	+ 560	− 9600	∽
c^2d^2e	.	.	.	.	+ 4080	− 17600
cd^4	.	.	.	.	.	+ 7680
	±245	±5070	±6550	±5500	±23010	±25475

	BCE^3	BD^2E^2	CD^2E^2	CD^3E	D^5
a^2f^3	− 5	+ 5	+ 5	− 5	+ 1
$abef^2$	+ 30	− 30	− 60	+ 130	− 25
$acdf^2$	− 220	+ 40	− 380	+ 380	− 80
ace^2f	− 40	− 1070	+ 640	− 380	+ 250
ad^2ef	+ 300	+ 940	− 1640	− 940	+ 360
ade^3	− 120	− 160	+ 620	− 800	− 750
b^2df^2	+ 400	− 40	+ 200	− 400	+ 100
b^2e^2f	− 185	+ 1145	− 25	− 125	∽
bc^2f^2	− 180	∽	+ 180	− 300	∽
$bcdef$	− 60	− 1020	+ 4380	+ 1140	− 1000
bce^3	+ 80	− 650	− 2900	+ 3500	∽
bd^3f	.	+ 320	+ 2000	+ 2080	− 960
bd^2e^2	.	+ 520	+ 100	+ 2600	+ 3000
c^3ef	.	.	− 3960	+ 1200	∽
c^2d^2f	.	.	− 480	− 960	+ 1600
c^2de^2	.	.	+ 1320	− 14800	∽
cd^3e	.	.	.	+ 7680	− 4800
d^5	.	.	.	.	+ 2304
	±810	±2970	±9445	±18710	±7615

	BDE^3	C^2E^3	CD^2E^2	D^4E
abf^3	∽	− 10	+ 20	− 5
$acef^2$	− 240	+ 40	− 20	+ 80
ad^2f^2	+ 120	− 300	− 20	+ 40
ade^2f	+ 140	+ 220	− 1190	− 20
ae^4	− 80	− 40	+ 400	− 500
b^2ef^2	+ 240	+ 50	− 100	+ 25
$bcdf^2$	− 120	+ 600	− 60	− 200
bce^2f	− 240	− 540	+ 1980	− 250
bd^2ef	+ 160	+ 500	+ 3160	− 320
bde^3	+ 20	− 200	− 850	+ 2750
c^3f^2	.	− 360	∽	∽
c^2def	.	− 120	− 4080	+ 800
c^2e^3	.	+ 160	− 2600	∽
cd^3f	.	.	+ 1280	+ 320
cd^2e^2	.	.	+ 2080	− 5600
d^4e	.	.	.	+ 2880
	± 680	±1570	±8920	±6895

	BE^4	CDE^3	D^3E^2
acf^3	− 20	∽	+ 10
$adef^2$	+ 30	− 320	+ 60
ae^3f	− 15	+ 140	− 340
b^2f^3	+ 20	∽	∽
$bcef^2$	− 30	+ 360	− 50
bd^2f^2	∽	+ 400	− 120
bde^2f	+ 20	+ 440	+ 1020
be^4	− 5	− 300	+ 500
c^2df^2	.	− 480	∽
c^2e^2f	.	− 960	+ 100
cd^2ef	.	+ 640	− 960
cde^3	.	+ 80	− 1900
d^4f	.	.	+ 640
d^3e^2	.	.	+ 1040
	±70	±2060	±3370

	CE^4	D^2E^3
adf^3	− 30	+ 10
ae^2f^2	+ 15	− 100
bcf^3	+ 30	∽
$bdef^2$	+ 100	+ 190
be^3f	− 55	+ 260
c^2ef^2	− 120	∽
cd^2f^2	∽	− 160
cde^2f	+ 80	− 540
ce^4	− 20	− 200
d^3ef	.	+ 480
d^2e^3	.	+ 60
	±225	±1000

	DE^4
aef^3	− 15
bdf^3	+ 20
be^2f^2	+ 55
c^2f^3	∽
$cdef^2$	− 80
ce^3f	− 70
d^3f^2	∽
d^2e^2f	+ 120
de^4	− 30
	±195

	E^5
af^4	− 1
bef^3	+ 5
cdf^3	∽
ce^2f^2	− 10
d^2ef^2	∽
de^3f	+ 10
e^5	− 4
	±15

Upon writing $(\mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E})=(x^3, xy^2, xy^2, y^3)$, the foregoing values of $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ become covariants of the quintic $(a, b, c, d, e, f \between x, y)^5$. In fact we have

$$\begin{aligned} \tfrac{1}{5}\mathfrak{C} &= 2(\text{Tab. No. } 15), \\ \tfrac{1}{5}\mathfrak{D} &= 2(\text{Tab. No. } 18), \\ \tfrac{1}{5}\mathfrak{E} &= (\text{Tab. No. } 13)^2(\text{Tab. No. } 14)-3(\text{Tab. No. } 15)^2, \\ \mathfrak{F} &= (\text{Tab. No. } 13)^2(\text{Tab. No. } 17)-2(\text{Tab. No. } 15)(\text{Tab. No. } 18), \end{aligned}$$

where the Tables referred to are those of my Fifth Memoir on Quantics, Tab. No. 13 being the quintic itself. This is a further verification of the foregoing result.

I will conclude by showing how the formula may be applied to the reduction of the general quintic equation to Mr. JERRARD's form $x^5+ax+b=0$. It was long ago remarked by Professor SYLVESTER that TSCHIRNHAUSEN's Transformation, in its original form, gave the means of effecting this reduction. In fact, if the transforming equation be

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4,$$

then the equation in y is of the form

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0,$$

where $\mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ are, in regard to $\alpha, \beta, \gamma, \delta, \varepsilon$, of the degrees 1, 2, 3, 4, 5 respectively. And by assuming, say α a linear function of $\beta, \gamma, \delta, \varepsilon$, we may make $\mathrm{B}=0$, and we have then $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F}$ functions of the degrees 2, 3, 4, 5 respectively of the quantities $\beta, \gamma, \delta, \varepsilon$: and these can be determined by means of a quadric equation and a cubic equation in such manner that $\mathfrak{C}=0$, $\mathfrak{D}=0$, in which case the equation in y will be of the required form. For considering $\beta, \gamma, \delta, \varepsilon$ as the coordinates of a point in space, the equations $\mathfrak{C}=0$, $\mathfrak{D}=0$ will be the equations of a quadric surface and a cubic surface respectively, and if $\beta, \gamma, \delta, \varepsilon$ be the coordinates of a point on the curve of intersection, the required conditions will be satisfied. And by combining with the equation of this the quadric surface, the equation of any tangent plane thereto (or by the different process which is made use of in the sequel), we may, by means of a quadric equation, find a generating line of the quadric surface, and then, by means of a cubic equation, a point of intersection of this line with the cubic surface, *i. e.* a point the coordinates whereof give the required values $\beta, \gamma, \delta, \varepsilon$. And similarly for the new form of TSCHIRNHAUSEN's Transformation; the only difference being that, starting with an equation in y which contains the four arbitrary quantities B, C, D, E, we obtain in the first instance an equation $(1, 0, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}, \mathfrak{F} \between y, 1)^5=0$ wanting the second term. And then B, C, D, E are to be so determined that $\mathfrak{C}=0$, $\mathfrak{D}=0$.

To proceed with the reduction, I write the foregoing value of $\mathfrak{C}$ in the form

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{llll} 4ac-4b^2, & 6ad-6bc, & 4ae-4bd, & af-be \\ 6ad-6bc, & 4ae+20bd-24c^2, & af+15be-16cd, & 4bf-4ce \\ 4ae-4bd, & af+15be-16cd, & 4bf+20ce-24d^2, & 6cf-6de \\ af-be, & 4bf-4ce, & 6cf-6de, & 4df-4e^2 \end{array}\right\between \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E})^2,$$

which for shortness may be represented by

$$\tfrac{2}{5}\mathfrak{C}=\left(\begin{array}{cccc} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & p \end{array}\right\rangle\!\!\left\langle B, C, D, E^2\right),$$

or say

$$\tfrac{2}{5}\mathfrak{C}=(\Omega \rangle\!\langle B, C, D, E)^2.$$

Now, by a formula in my memoir "On the Automorphic Linear Transformation of a Bipartite Quadric Function*," if Υ denote any skew symmetric matrix of the order 4, then if

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\Omega \rangle\!\langle B', C', D', E'),$$

in which formula Ω^{-1}, $\Omega-\Upsilon$, $(\Omega+\Upsilon)^{-1}$, Ω are all matrices which are to be compounded together into a single matrix, we have identically

$$(\Omega \rangle\!\langle B, C, D, E)^2=(\Omega \rangle\!\langle B', C', D', E')^2.$$

Let Q denote the determinant $|\Omega+\Upsilon|$, then the terms of the matrix $(\Omega+\Upsilon)^{-1}$ are respectively divided by Q, and we may write

$$(\Omega+\Upsilon)^{-1}=\frac{1}{Q}.\,Q(\Omega+\Upsilon)^{-1},$$

where $Q(\Omega+\Upsilon)^{-1}$ is the matrix obtained from the matrix $(\Omega+\Upsilon)^{-1}$ by multiplying each term by Q, the terms of $Q(\Omega+\Upsilon)^{-1}$ being thus rational and integral functions of the terms of the matrix $(\Omega+\Upsilon)$. Hence if, instead of the before-mentioned relation between (B, C, D, E) and (B′, C′, D′, E′), we assume

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega \rangle\!\langle B', C', D', E'),$$

we find

$$(\Omega \rangle\!\langle B, C, D, E)^2=Q^2(\Omega \rangle\!\langle B', C', D', E')^2.$$

And if the matrix Υ is such that we have $Q=0$, *i. e.* Det. $(\Omega+\Upsilon)=0$ (which is a quadric relation between the terms of the skew matrix, that is, each term is contained therein in the first and second powers only), then the equation becomes

$$(\Omega \rangle\!\langle B, C, D, E)^2=0.$$

It is clear that this can only be the case in consequence of the coefficients of transformation in the equation

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega \rangle\!\langle B', C', D', E')$$

being such that there shall exist at least two linear relations between the quantities (B, C, D, E), and I assume (without stopping to prove it) that they are such that the number of such linear relations is in fact two. That is, the last-mentioned equation establishes between the quantities (B, C, D, E) two linear relations, in virtue whereof

* Philosophical Transactions, vol. cxlviii. (1858), see p. 44.

$\mathfrak{C}=0$. And this being so, we may, without loss of generality, write $D'=0$, $E'=0$, or put

$$(B, C, D, E)=(\Omega^{-1}(\Omega-\Upsilon)Q(\Omega+\Upsilon)^{-1}\Omega \between B', C', 0, 0);$$

so that B, C, D, E are linear functions of B', C', such that $\mathfrak{C}=0$. And then substituting these values for (B, C, D, E), we find $\mathfrak{D}$ a cubic function of B', C'; so that, putting $\mathfrak{D}=0$, we have a cubic equation to determine the ratio B' : C'.

The foregoing reasoning presents no real difficulty, but it is expressed by means of a very condensed notation, and it may be proper to illustrate it by the case of the quadric function $x^2+y^2+z^2$. Considering the equations

$$\begin{aligned} x&=(1+\lambda^2-\mu^2-\nu^2)x'+ & &2(\lambda\mu-\nu)y'+ & &2(\lambda\nu+\mu)z', \\ y&= 2(\lambda\mu+\nu)x'+ & &(1-\lambda^2+\mu^2-\nu^2)y'+ & &2(\mu\nu-\lambda)z', \\ z&= 2(\nu\lambda-\mu)x'+ & &2(\mu\nu+\lambda)y'+ & &(1-\lambda^2-\mu^2+\nu^2)z', \end{aligned}$$

these equations, if the expressions for x, y, z had been divided by $1+\lambda^2+\mu^2+\nu^2$, would have given

$$x^2+y^2+z^2=x'^2+y'^2+z'^2.$$

Hence they actually do give

$$x^2+y^2+z^2=(1+\lambda^2+\mu^2+\nu^2)^2(x'^2+y'^2+z'^2);$$

or if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

they give

$$x^2+y^2+z^2=0.$$

But if

$$1+\lambda^2+\mu^2+\nu^2=0,$$

then

$$\begin{aligned} &1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu-\nu) : 2(\lambda\nu+\mu) \\ =&2(\lambda\mu+\nu) : 1-\lambda^2+\mu^2-\nu^2 : 2(\mu\nu-\lambda) \\ =&2(\nu\lambda-\mu) : 2(\mu\nu+\lambda) : 1-\lambda^2-\mu^2+\nu^2; \end{aligned}$$

so that we have

$$x : y : z=1+\lambda^2-\mu^2-\nu^2 : 2(\lambda\mu+\nu) : 2(\nu\lambda-\mu),$$

which is the same result as would have been found by writing $y'=z'=0$, and which comes to saying that x, y, z are not independent, but are connected by two linear relations.

The equation Det. $(\Omega+\Upsilon)=0$, written at length, will be

$$\begin{vmatrix} a, & h-\tau, & g+\sigma, & l+\lambda \\ h+\tau, & b, & f-\rho, & m+\mu \\ g-\sigma, & f+\rho, & c, & n+\nu \\ l-\lambda, & m-\mu, & n-\nu, & p \end{vmatrix}=0,$$

where $\lambda, \mu, \nu, \rho, \sigma, \tau$ are the arbitrary constituents of the skew matrix; or developing, this is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & p \end{vmatrix}$$

$$+\left(\begin{array}{cccccc} bc-f^2, & fg-ch, & hf-bg, & mg-nh, & fm-bn, & -fn+cm \\ fg-ch, & ca-g^2, & gh-af, & -gl+an, & nh-lf, & gn-cl \\ hf-bg, & gh-af, & ab-h^2, & hl-am, & -hm+bl, & lf-mg \\ mg-nh, & fm-bn, & -fn+cm, & ap-l^2, & ph-lm, & pg-ln \\ -gl+an, & nh-lf, & gn-cl, & ph-lm, & bp-m^2, & pf-mn \\ hl-am, & -hm+bl, & lf-mg, & pg-ln, & pf-mn, & cp-n^2 \end{array}\right)(\lambda, \mu, \nu, \varrho, \sigma, \tau)^2$$

$$+(\lambda\varrho+\mu\sigma+\nu\tau)^2=0,$$

the first term whereof, substituting for (a, b, c, f, g, h, l, m, n, p) their values, is in fact equal to the discriminant a^4f^4+ &c. of the quintic $(a, b, c, d, e, f)(X, Y)^5$. There is no loss of generality in putting all but two of the quantities $(\lambda, \mu, \nu, \varrho, \sigma, \tau)$ equal to zero; in fact this leaves in the formulæ a single arbitrary quantity, which is the right number, since the ratios B : C : D : E have to satisfy only the two conditions $\mathfrak{C}=0$, $\mathfrak{D}=0$.

Addition, Nov. 10, 1862.

I take the opportunity of remarking, with reference to my memoir "On a New Auxiliary Equation in the Theory of Equations of the Fifth Order*," that I recently discovered that the auxiliary equation there considered is in fact due to Jacobi, who, in his paper, "Observatiunculæ ad theoriam æquationum pertinentes†," under the heading "Observatio de æquatione sexti gradus ad quam æquationes quinti gradus revocari possunt," gives the theory, and observes that the equation is of the form

$$\phi^6+a_2\phi^4+a_4\phi^2+a_6=32\sqrt{\square}\,\phi,$$

and mentions that the value of a_2 is easily found to be (I adapt his notation to the denumerate form $(a, b, c, d, e, f)(v, 1)^5=0$)

$$=40ae-16bd+6c^2$$

(this ought, however, to be divided by -2), but that the values of a_4, a_6 "paullo ampliores calculos poscunt."

The value of the coefficient in question is correctly obtained (page 270 of my memoir) in the form

$$\begin{array}{l} -c^2 \\ +2(-16ae+4bd-c^2) \\ +12ae; \end{array}$$

but the reduced value is given in two places (page 271) as equal to

$$\begin{array}{lcl} -32\,ae, & \text{this should be} & -20\,ae, \\ +\ 8\,bd, & ,, & +\ 8\,bd, \\ -\ 3c^2, & ,, & -\ 3c^2. \end{array}$$

The last-mentioned correct value was used in obtaining the coefficient for the standard form, which coefficient is given correctly, page 274.

* Philosophical Transactions, vol. cli. (1861) pp. 263–276.

† Crelle, t. xiii. (1835) pp. 340–352.